Steady, Gravity-Driven Water Flow in a Full Pipe of Constant Cross Section

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The Problem

We consider the problem of water flow in a pipe of constant cross-sectional area due to an elevation change between the pipe inlet and outlet. The flow is assumed to be steady and the pipe flows full at all cross sections. The application is the design of a single leg of a water distribution network that is fed by one reservoir (the “source”). There are many references for this type of design (The Blue Book). The present work is not intended to be a thorough, comprehensive treatment of the subject matter. It is, however, an attempt to provide a link between the subject of fluid mechanics learned in a typical undergraduate mechanical engineering curriculum, with which the students should already be familiar, and the more-comprehensive coverage of this topic in a book such as The Blue Book.

Another purpose of this work is to address the fundamental problem of pipe sizing for a few simple single-pipe gravity-driven water systems, and to begin to address some of the other critical design issues pertinent to gravity-driven water distribution network.

The fundamental problem of designing a gravity-driven water distribution network is normally stated as follows. For the required volume flow rate of water to be delivered to an end use, and given dimensions of the site (elevation of reservoir, run of pipe, contour of the land, etc.), calculate the pipe diameters for the network that satisfy these conditions. Also included in the design will be sizing and location intermediate water storage tanks, the positions and types of valves and fittings, and the serious consideration of issues like the elimination of possible undesirable vacuum conditions along the water flow path that may lead to pipe-wall collapse.

The Energy Equation for Flow in a Pipe: A Review of Fluid Dynamics

We begin with the mechanical energy equation for steady flow in a round pipe. This equation is covered in a typical undergraduate course in fluid mechanics. The pipe diameter is $D$, the cross-sectional average flow speed in the pipe is $u$, and the water has density $\rho$. The static pressure$^1$ at any location in the pipe is $p$, and the elevation measured from the lowest point in the pipe is a vertical coordinate $z$ (see Fig. 1). The mechanical energy equation is sometimes referred to as a “modified Bernoulli equation” because it looks like the Bernoulli equation. In contrast to the Bernoulli equation, which is only valid for inviscid flow (i.e., frictionless flow), included in the

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$^1$Recall that the static pressure is the pressure that would be measured in a flowing fluid with no effect from the flow speed. It could be measured by a tap installed in the pipe wall so that the opening of the tap is parallel to the motion of the fluid.
modified Bernoulli equation is a term that accounts for energy loss due to friction. This term is referred to as a “head loss”\(^2\). Thus, the modified Bernoulli equation equates the change in total mechanical energy between the pipe inlet and outlet to the energy loss in the pipe due to friction.

There are two classifications for the energy loss in the pipe. The first is the energy loss due to shear stress between the moving fluid and the stationary pipe wall (“major loss”). The second is the energy loss due to pipe fittings, valves, contractions, and expansions in the flow path and is referred to as a “minor loss.” Referring to Fig. 1, the modified Bernoulli equation is (Cengel and Cimbala, 2006a)

\[
(p_1 \rho + \alpha_1 \frac{\pi_1^2}{2} + gz_1) - (p_2 \rho + \alpha_2 \frac{\pi_2^2}{2} + gz_2) = H_L = [f \left( \frac{L}{D} \right) + \sum_{i=1}^{M} L_i \left| \frac{\pi_i}{D} \right| + \sum_{i=1}^{N} K_i \frac{\pi_i^2}{2}],
\]

where state 1 is at the source and state 2 is the delivery point. The terms in each parenthesis on the left side of eqn. (1) account for pressure energy, kinetic energy, and potential energy, all per unit mass flow rate of fluid. The term on the right side of eqn. (1) is the energy loss due to friction. It is an extended form of the classical “Darcy-Weisbach” equation,

\[
H_L = f \frac{L \pi^2}{D^2},
\]

where terms for the minor loss have been included. The first term on the right side of eqn. (1) is the major loss and the second and third terms account for the minor loss. The minor loss can be characterized by a dimensionless loss coefficient \(K\) which, when summed over all minor-loss elements \((N)\), accounts for the total minor loss along the pipe flow path. For example, the minor loss coefficient for flow entering a pipe protruding through the wall of a reservoir (this condition is termed a “re-entry”) is about \(K = 0.78\) (Fox and McDonald, 1992a). Minor loss coefficients can be found in your fluid mechanics textbook.

There is an alternate way of writing the minor loss. In eqn. (1), \(L_i\) is called the “equivalent length.” In the equivalent-length method, we calculate the energy loss for a pipe fitting or similar

\(^2\)Head can be expressed as a height (given the symbol \(h\) here) or as the product of height and gravity constant, \(g\), (given the symbol \(H\)).
minor-loss element as if it occurs in a number of pipe-diameters of straight pipe. This method is commonly used for certain fittings. For example, the minor loss for an elbow is often accounted for using the equivalent-length method. For a standard 90° elbow, the minor loss is from $L_e/D = 30$ (Fox and McDonald, 1992b). In eqn. (1), the total minor loss for those elements using the equivalent length method is from the sum of $M$ total elements.

We will assume that the pressure is to be measured from the local atmospheric value, so that at the reservoir surface (state 1) $p_1 = 0$ by definition. Also at the surface of the reservoir, $u_1 = 0$. In addition, since we define the coordinate $z$ to be measured from the lowest point in the pipe, $z_2 = 0$. With these developments, eqn. (1) simplifies to

$$1 - \frac{p_2}{\rho g z_1} = [f(\pi, D)\left(\frac{L}{z_1} + \frac{D}{z_1} \sum_{i=1}^{M} L_e D_i\right) + \frac{D}{z_1} \left(\alpha_2 + \sum_{i=1}^{N} K_i\right) + \frac{Dz_1}{2gD}]$$

(3)

For turbulent flow, $\alpha_2 \approx 1.05$ and for laminar flow, $\alpha_2 = 2$ (Cengel and Cimbala, 2006b).

The term $f$ in eqs. (1)-(3) is the Moody friction factor for energy loss in a straight pipe. We write it as $f(\pi, D)$ to keep in mind that the friction factor depends on the mean flow speed in the pipe and the pipe diameter through the Reynolds number. Typical values for $f(\pi, D)$ range from 0.05 to 0.01 for turbulent flow from the near-transition regime to highly turbulent, respectively. The values of $f(\pi, D)$ are determined from different formulas for laminar and turbulent flow. These are (White, 1999a)

$$f(\pi, D) = \begin{cases} 64/Re, & \text{laminar flow, } Re < 2100 \\ -2 \log_{10}\left[\frac{2.51}{Re^{0.5}} + \frac{e}{D^{0.5}}\right]^{-2}, & \text{turbulent flow} \end{cases}$$

(4)

where $Re$ is the Reynolds number,

$$Re = \frac{\pi D}{\nu},$$

(5)

which clearly depends on $\pi$ and $D$ (but is never written as $Re(\pi, D)$) and $\nu$ is the kinematic viscosity of water at ambient temperature. The value for $\nu$ can be looked up in the appendix of your fluids or heat transfer textbooks. From your fluid mechanics course, you will recall that the flow is laminar for $Re < 2100$ and turbulent otherwise. For small values of $z_1$ and small $D$, the flow in the pipe may indeed be laminar even though for most cases turbulent flow is common. The explicit formula for $f(\pi, D)$ from eqn. (4) for turbulent flow is referred to as the Colebrook formula and is the accepted design formula for $f$ for turbulent flow in pipes. Note that it is implicit in $f(\pi, D)$ and requires iteration to solve. An explicit approximation, say, within ±3%, to the Colebrook equation is

$$f(\pi, D) = \begin{cases} 64/Re, & \text{laminar flow, } Re < 2100 \\ -1.8 \log_{10}\left[\frac{6.9}{Re^{0.5}} + \frac{e}{D}^{1.11}\right]^{-2}, & \text{turbulent flow} \end{cases}$$

(6)

which is from Haaland (White, 1999a), and is easier to implement. For the results presented below, we use a nearly exact curve-fit to friction-factor data from Churchill (1977) which is equivalent to the more-accurate Colebrook equation.

In eqn. (4) the term $e$ is the absolute roughness of the pipe wall. For PVC irrigation pipe, $e \approx 5 \times 10^{-6}$ ft ±60% (White, 1999b). For galvanized steel pipe, $e$ is about 100 times larger than this value.

It is interesting to note that the ratio $L/z_1$ in eqn. (3) contains, as a single term, the two most important lengths associated with a gravity-driven water system. These are the elevation of the source, $z_1$, and the length of the pipe, $L$. To illustrate the importance of this term and to present
some numerical results for the pipe diameter needed for a prescribed volume flow rate of water, we will consider the following two simple cases. The first is an ideal case where the pipe is straight (i.e., no bends or curves in the pipe), and the second is for a case where the pipe is of arbitrary length.

At this point, it is worthwhile to compare the present case of gravity-driven flow in a pipe with the cases that are normally considered for pipe flow in a typical undergraduate course in fluid mechanics. The energy equation, eqn. (1), is obviously the same for all cases, however, in a course in undergraduate fluid mechanics, four classes of problems for flow in a single pipe are typically considered. These are

1. $L, Q$, and $D$ known, $p_2 - p_1$ unknown,
2. $p_2 - p_1, Q$, and $D$ known, $L$ unknown,
3. $p_2 - p_1, L$, and $D$ known, $Q$ unknown, and
4. $p_2 - p_1, L$, and $Q$ known, $D$ unknown.

The first two classes are particularly simple because with both $Q$ and $D$ known, the flow speed, $u$, is easily calculated from the continuity equation,

$$Q = \pi A = \pi \pi D^2 / 4,$$

and the remaining unknown, either $L$ or $p_2 - p_1$, easily found from application of eqn. (1). The last two classes are slightly more challenging since, with either $Q$ or $D$ unknown, the Reynolds number is not known so that $f(u, D)$ is also not known. Either a pencil-and-paper iterative method is needed to solve eqn. (1) for $u$ or a root-finder in a program like Mathcad or Excel may be used to solve eqn. (1) for $u$. The present work is a special case of class 4 where the inlet static pressure, $p_1$, is zero. Thus, you may have already solved a problem similar to that being considered in this work in your fluid mechanics course. However, in most examples from your fluids class, the focus tends to be on systems where the flow is driven by a pump (i.e., $p_1 \neq 0$) and particular attention is payed to the calculation of the major and minor losses. In fact, the elevation change in the typical example problem is most often neglected. Of course, this effect is the only driving force for flow in the present work.

**Straight Pipe Case**

As illustrated in Fig. 1, if the pipe is straight, the pipe length $L$ is related to the elevation, $z_1$, and the horizontal run of the pipe, $\ell$ (= distance between the inlet and outlet of the pipe measured in the horizontal plane), through the Pythagorean theorem. Thus,

$$L = z_1 \sqrt{1 + (\ell / z_1)^2} = z_1 \sqrt{1 + s^2},$$

where $s$ is the slope of the pipe (rise/run) or $s = z_1 / \ell$. $z_1$ may be determined by an instrument such as a GPS or an altimeter, and the run dimension $\ell$ from a map or extracted from a tape measurement of the site. With eqn. (8), eqn. (3) becomes

$$1 - \frac{p_2}{\rho g z_1} - \left[ f(\pi, D)(\sqrt{1 + s^2} + \frac{D}{z_1} \sum_{i=1}^{M} L_i |_{i=1}^{M} + \frac{D}{z_1} (\alpha_2 + \sum_{i=1}^{N} k_i)) \right] \frac{\pi^2}{2 g D} = 0. $$

Therefore, at the most basic level, we see that the slope, $s$, of a water delivery pipe in a single pipe, gravity-driven system is the key controlling parameter in the design. The problem of
generating design graphs using eqn. (9) can now be stated as follows. For given values of the slope \((s)\) of the pipe, pipe diameter \((D)\), and total minor loss coefficients \((K_i\) and \(\frac{L_e D}{D_i}\)) calculate the mean flow speed in the pipe \((\bar{u})\) by solving the non-linear algebraic equation of eqn. (9). From the solution for \(\bar{u}\), the volume flow rate of water, \(Q\), is simply determined from the continuity equation, eqn. (7).

The solution for \(\bar{u}\) from eqn. (9) can be carried out for many different values of \(s\) and \(D\) (and \(K_i\) and \(\frac{L_e D}{D_i}\)) and a series of curves produced for \(Q\) as a function of \(s\) for fixed values of \(D\). These would be useful to provide a first-cut prediction for the volume flow rate produced for a single-pipe gravity water system of a given pipe diameter and slope (and \(K_i\) and \(\frac{L_e D}{D_i}\)).

Before we carry out this solution, two comments are needed. First, for all systems the ratio \(\frac{D}{z_1} \ll 1\). Thus the minor losses and \(\alpha_2\) in eqn. (9) may be neglected as a first approximation. We can assess the effect of these terms on the flow speed later on, but for simplicity we will neglect the minor loss and \(\alpha_2\) terms for now. Secondly, the term \(p_2/\rho g z_1\) in eqn. (9) is the ratio of the delivery static pressure (at the low end of the pipe) to the hydrostatic pressure of the reservoir\(^3\). This ratio is a design parameter which we are free to adjust to our needs. If the entire water distribution system is considered, i.e., from the source to the final delivery point where water is delivered through a faucet valve, then \(p_2 = 0\) or \(p_2/\rho g z_1 = 0\). This is the case for which the water flows from the end of the pipe only due to its kinetic energy and has no assistance from static pressure at that point. At the other extreme, for \(p_2/\rho g z_1 = 1\), the pressure at the delivery point is purely hydrostatic and there is no flow in the system. The normal case would be between these extremes where there is a need to distribute water from the delivery point to houses in a village far from this point. This requires \(p_2\) to be greater than zero. In the design charts presented below, we will label the term \(p_2/\rho g z_1\) as \(F\) and allow \(F\) to take on certain values like 0, 0.1, 0.25, 0.50, etc. In this way we will account for a range of delivery pressures at the end of the pipe. Below, we will extend the ideas presented here to include multiple pipes.

With the above developments, the final form of the energy equation to be solved for \(\bar{u}\) for this case is

\[
1 - F - [f(\bar{u}, D)\left(\sqrt{1 + s^{-2}} + \frac{D}{z_1} \sum_{i=1}^{M} L_i \right) + \frac{D}{z_1} \left(\alpha_2 + \sum_{i=1}^{N} K_i\right)] \frac{\bar{u}^2}{2gD} = 0,
\]

\[
(10)
\]

where the terms involving \(\frac{D}{z_1}\) are to be neglected at first.

**Case for Pipe of Arbitrary Length**

In this case, the pipe length, \(L\), is not tied to the elevation of the reservoir or the run, but is arbitrary. The length may consist of straight pipe and fittings including elbows, etc., or it may be bent in a curved manner to conform to the contour of the earth in which it is buried. In this case \(L\) may be written as

\[
L > \sqrt{z_1^2 + \ell^2} = z_1 \sqrt{1 + s^{-2}},
\]

\[
(11)
\]

or

\[
L = \lambda \sqrt{z_1^2 + \ell^2} = \lambda z_1 \sqrt{1 + s^{-2}},
\]

\[
(12)
\]

where \(\lambda\) is a dimensionless number > 1 defined as

\[
\lambda = \frac{L}{\sqrt{z_1^2 + \ell^2}} = \frac{L}{z_1 \sqrt{1 + s^{-2}}}.
\]

\[
(13)
\]

\(^3\)Recall that the hydrostatic pressure, \(p_2/\rho g z_1\), is that caused by a static head of fluid, \(z_1\); please do not confuse this with the “static” pressure which is the pressure measured in a fluid when it is in motion.
For this case, the final form of the energy equation to be solved for $\pi$ is

$$1 - F - \left[ f(\pi, D) (\lambda \sqrt{1 + s^2} + \frac{D}{z_1} \sum_{i=1}^{M} \frac{L_i}{D_i}) + \frac{D}{z_1} (\alpha_2 + \sum_{i=1}^{N} K_i) \right] \frac{\pi^2}{2gD} = 0, \quad (14)$$

where the terms involving $D/z_1$ are to be neglected at first.

**Method of Solution: Minor-Lossless Flow**

From our inspection of eqs. (10) and (14), we see that eqn. (10) is a special case of the more-general eqn. (14) with $\lambda$ set equal to one. Thus, eqn. (14) is the energy equation of interest. The value for $\lambda$ may be set to any value $\geq 1$ that the designer specifies. The solution of eqn. (14) is carried out in [Mathcad](http://www.mathcad.com) using the [root](http://www.mathcad.com) function. The friction factor is from eqn. (4). Alternately, the commercial code [Matlab](http://www.mathworks.com) and a continuous smooth curve-fit to the laminar and turbulent friction factor from Churchill (1977) can be used to provide similar results.

**Results: Design Graphs for Minor-Lossless Flow**

We first present a graph of $Q$ from the solution of eqn. (14) (eqn. (7) is used to calculate $Q$ from $\pi$) for two pipe diameters and two values for pipe length (see Fig. 2). We see that the water flow rate increases with slope and with pipe diameter. This can be explained from an intuitive argument. Imagine a single, straight, open-ended pipe held in your hands with a constant source of water at the top opening. If the slope of the pipe is zero (horizontal) there is no effect from gravity pulling the water downward because of the zero slope and, thus, the flow rate is zero (see how all of the curves in Fig. 2 tend to zero flow rate as the slope, $s$, approaches zero). As the slope of the pipe increases, gravity exerts a greater pull on the water and the flow rate increases. If the pipe has a slope of one (the pipe inclined at a $45^\circ$ angle), the flow rate increases to the largest value seen in this figure.

As the slope of the pipe approaches infinity (i.e., a vertical pipe), the pipe length becomes equal to the elevation of the top of the pipe and the water flow rate reaches a maximum value. This is seen in Fig. 3, which is identical to Fig. 2 except that the slope axis now ranges from 0.2 to 10. The maximum flow rate of water in this case can be compared with the classical “terminal velocity” of a body falling in a fluid under its own weight. In this situation, the speed of the fall is such that the drag acting on the body is exactly balanced by the body’s weight. The acceleration is zero and the speed that the body achieves under this condition is referred to as “terminal.” Thus, with a large slope (10 is large enough to be considered infinite), the water is simply free-falling vertically in the pipe and will reach a speed where the weight of the water is exactly balanced by the friction force at the pipe wall. Under these conditions the energy equation to be solved for the terminal velocity of the water is from eqn. (9) with $s \to \infty$. With minor losses neglected and the delivery pressure of zero, we get

$$1 - f(\pi, D) \frac{\pi^2}{2gD} = 0. \quad (15)$$

The solution of eqn. (15), with the flow speed, $\pi$, converted to volume flow rate, $Q$, appears in Fig. 4 for a broad range of pipe diameter.

Consider the following simple numerical example. For a 1-inch nominal PVC pipe ($D = 1.049$ in), the final step in the root-finding solution of eqn. (15) gives $f(\pi, D) = 0.0209$. The solution
Figure 2: Volume flow rate of water vs. mean slope of pipe for two different pipe diameters and two values for pipe length. \( \lambda = 1 \) corresponds to the straight pipe case. \( \lambda = 1.5 \) is for a pipe length 50% greater than the straight pipe case. Dimensionless delivery pressure corresponds to \( F = 0.5 \). No minor loss.

Figure 3: Same as Fig. 2 except slope axis ranges from 0.2 to 10. Terminal values for \( Q \) are seen in this figure as slope \( \rightarrow \infty \).
for $\tau$ from eqn. (15) gives

$$\tau = \sqrt{\frac{2gD}{f(\tau, D)}} = \sqrt{\frac{2 \cdot 9.81 \text{ m/s}^2 \cdot (1.049/39.372) \text{ m}}{0.0209}} = 4.205 \text{ m/s.}$$

(16)

The volume flow rate for this example is from eqn. (7)

$$Q = \tau \pi D^2/4 = 4.205 \text{ m/s} \cdot \pi \cdot ((1.049/39.372) \text{ m})^2/4 = 0.00234 \text{ m}^3/\text{s} = 2.34 \text{ l/s.}$$

(17)

While this limiting case is of interest from a theoretical perspective and as an upper-bound on the volume flow rate of water in a gravity-driven flow, it has little relevance in an actual design since few, if any, reservoirs are located vertically or near-vertically above the delivery spot.

Water flow rate also increases with pipe diameter as seen in Fig. 2. For a given length of pipe, the frictional energy loss (which comes from shear between the pipe wall and water) is proportional to the circumference of the pipe ($\pi D$) and the flow rate is dependent on the pipe cross sectional area ($\pi D^2/4$). The ratio of the two is proportional to $D$ so that, as $D$ increases, more water can pass through the cross section of the pipe per unit of shear stress at the pipe wall. We also see in Fig. 2 that, as expected, the water flow rate decreases as the pipe gets longer because of the additional pressure drop due to friction.

Several design graphs are presented in the figures below in Figs. 5-12. The plots are of $Q$ as a function of the mean slope, $s$, for a range of PVC nominal pipe diameters (the actual inside diameters are slightly larger than the nominal size and were used as $D$ in the calculations), and for different values for $F$, $\lambda$, and, later on, $K_i D/z_1$. These plots are similar to that of Fig. 2 except that the design plots are presented with log-log axes to be able to better read the numbers for $s$ and $Q$ over a large range of values.
Figure 5: Volume flow rate of water vs. mean slope of pipe for five nominal PVC-pipe diameters. Delivery pressure corresponds to $F = 0$ and $\lambda = 1$ (straight pipe case). No minor losses. Laminar flow is evident for the smallest pipe diameter at the smallest slopes.

Figure 6: Volume flow rate of water vs. mean slope of pipe for five nominal PVC-pipe diameters. Delivery pressure corresponds to $F = 0.1$ and $\lambda = 1$ (straight pipe case). No minor losses. Laminar flow is evident for the smallest pipe diameter at the smallest slopes.
Figure 7: Volume flow rate of water vs. mean slope of pipe for five nominal PVC-pipe diameters. Delivery pressure corresponds to $F = 0.25$ and $\lambda = 1$ (straight pipe case). No minor losses. Laminar flow is evident for the smallest pipe diameter at the smallest slopes.

Figure 8: Volume flow rate of water vs. mean slope of pipe for five nominal PVC-pipe diameters. Delivery pressure corresponds to $F = 0.50$ and $\lambda = 1$ (straight pipe case). No minor losses. Laminar flow is evident for the smallest pipe diameter at the smallest slopes.
Figure 9: Volume flow rate of water vs. mean slope of pipe for five nominal PVC-pipe diameters. Delivery pressure corresponds to $F = 0$ and $\lambda = 1.5$ (50% longer than straight pipe case). No minor losses. Laminar flow is evident for the smallest pipe diameter at the smallest slopes.

Figure 10: Volume flow rate of water vs. mean slope of pipe for five nominal PVC-pipe diameters. Delivery pressure corresponds to $F = 0.1$ and $\lambda = 1.5$. No minor losses. Laminar flow is evident for the smallest pipe diameter at the smallest slopes.
Figure 11: Volume flow rate of water vs. mean slope of pipe for five nominal PVC-pipe diameters. Delivery pressure corresponds to $F = 0.25$ and $\lambda = 1.5$ (50% longer than straight pipe case). No minor losses. Laminar flow is evident for the smallest pipe diameter at the smallest slopes.

Figure 12: Volume flow rate of water vs. mean slope of pipe for five nominal PVC-pipe diameters. Delivery pressure corresponds to $F = 0.5$ and $\lambda = 1.5$ (50% longer than straight pipe case). No minor losses. Laminar flow is evident for the smallest pipe diameter at the smallest slopes.
Figure 13: Volume flow rate of water vs. mean slope of pipe for five nominal galvanized steel pipe diameters. Delivery pressure corresponds to \( F = 0 \) and \( \lambda = 1 \) (straight pipe case). No minor losses. Laminar flow is evident for the smallest pipe diameter at the smallest slopes.

Generally, all figures show that the volume flow rate increases in proportion to nearly the square root of \( s \) for a given \( D \). Increasing \( D \) also increases \( Q \), as discussed above, with the largest changes from size-to-size in the smallest diameter range. Increasing the pipe length by 50\% over the straight-pipe case of \( \lambda = 1 \) (Figs. 5-8) decreases \( Q \) by about 25\% (see Figs. 9-12 for \( \lambda = 1.5 \)), which is not a very large impact on the design. In Fig. 13, for galvanized straight steel pipe and delivery pressure of zero, we see that the water flow rate is only fractionally smaller than for the smoother PVC pipe (compare with Fig. 5).

As a final note, a Mathcad program has been produced that solves for the nominal pipe diameter for prescribed values of \( Q, s, \lambda, \) and \( F \) and includes the effects of minor losses. This code is available to the students. Please contact the author (gerard.jones@villanova.edu) for a copy. Compared with the above design graphs, the program has the advantages of including the minor losses and, with some simple modifications, can include multiple-pipe systems where the inlet pressure may not be zero for one or more of the pipes.

In all of the figures presented in this section, minor losses have been neglected. Below, we will consider the impact of minor losses on a design.

The Effects of Minor Losses

Minor losses, embodied by the terms \( K \) and \( L_e/D \) in eqn. (14), enter into the design problem as an energy loss added to that due to the straight pipe. There is a subtle difference between the major and minor losses. The major loss is, by its nature, one that is uniformly distributed along the pipe
length. In contrast, the minor losses occur at discrete locations along the flow path. By acting at discrete locations, minor losses impose a localized effect on the static pressure in the flow. For example, the minor loss associated with a partial blockage in a pipe flow will cause a reduction in static pressure at the blockage location and immediately downstream. Should the pressure fall to too low a value (i.e., a vacuum), there may be a localized collapse of the PVC pipe wall. This is obviously undesirable.

From a modeling perspective, to investigate the effect of the minor loss, one must consider a localized model of the pipe flow rather than a “lumped” type of model (one that considers just an entering state and an exiting state) that is represented by eqn. (14). We pursue this formulation now.

For the case of a straight pipe, the modified Bernoulli equation of eqn. (1) is written in differential form and integrated between points $z_1$ and any location along the length of the pipe at elevation $z$ to get the static pressure at that location as

$$p(z) = \frac{1}{\rho g z_1} - \frac{\pi^2}{2 g D} \left\{ f(\bar{\pi}, D) \left( 1 - \frac{z}{z_1} \right) \sqrt{1 + s^{-2}} + \frac{D}{z_1} \left[ \alpha_2 + \int_{z_1}^{z} dK(z) \right] \right\}. \quad (18)$$

The integral in eqn. (18) can be thought of as a running sum of the minor loss coefficient values between the top of the pipe (at $z = z_1$) and the location at any elevation $z$ along the pipe. These need to be specified by the system designer. Also note in eqn. (18) that we have opted to include the effects of the minor loss using the $K$ coefficients alone, i.e., any equivalent-length-type coefficients, if they exist, have been converted to $K$-type.

From our inspection of eqn. (18), we see that the dimensionless pressure distribution in the pipe, $p(z)/\rho g z_1$, depends on the dimensionless elevation of the pipe, $z/z_1$, the slope, $s$, pipe diameter $D$, and reservoir elevation, $z_1$, as well as the distribution and size of the $K$ coefficients. The value for $\bar{\pi}$ in eqn. (18) comes from the solution of eqn. (10), where $\sum_{i=1}^{N} K_i = \int_{z_1}^{0} dK(z)$ is the sum of all minor losses in the pipe system.

There is no simple way to represent the solution of eqn. (18) in the form of a few graphs as done above where we neglected minor losses. Instead, we will consider a few cases of interest and attempt to draw some general conclusions from these.

We consider a single straight pipe with a re-entrant loss or larger at the inlet (the end of the pipe protrudes into the reservoir) at $z = z_1$ where $K = K_1 = K_{entry}$ and two other minor losses. One is at $z = 0.5 z_1$ where $K = K_2 = 50$ and the other is at $z = 0$ where $K = K_3 = 100$. The sizes of the last two $K$ values correspond to partially open globe valves. Normally, there is a filter installed at the pipe inlet to prevent the entry of dirt and debris that could, over time, plug the system. To model this effect, we consider 20 and 200 times the re-entry $K = K_{entry}$ value of 0.78; the factor 20 corresponds to a small filter blockage, and 200, a larger one. We consider two nominal PVC pipe diameters of 2 inch and 0.75 inch, and assume a slope, $s$, of 1% and 10%. The case that will produce the lowest pressures in the pipe (i.e., worst case) is one where the delivery pressure $p_2$ is zero. We assume this value here to produce the most conservative results relative to this parameter. We also take $z_1 = 50$ m. The results show not much sensitivity to $z_1$ but large sensitivity to $s$ and $D$, as well as the $K$ distribution.

The local static pressure distribution is shown in Figs. 14 and 15 for the small and large filter blockage. Both figures show a sharp reduction in static pressure immediately after the flow leaves the reservoir at $z/z_1 = 1$. As the flow moves down the pipe toward smaller $z$ the pressure increases due to the fact that the pressure gain due to the decrease in potential energy is more than the frictional energy loss per unit of pipe flow path. The pressure falls suddenly due to the
Figure 14: Minor loss effects on static pressure in a single straight pipe. Small blockage at entrance to pipe in reservoir. $z/z_1 = 1$ is located at the reservoir. $z/z_1 = 0$ is at the delivery location.

minor loss at $z/z_1 = 0.5$. Between this location and the bottom of the pipe, the pressure rises again and then falls suddenly due to the minor loss at $z = 0$. Clearly, the main region of concern in the pipe is immediately downstream from the reservoir where the pressure is minimal. For the small blockage, there is slight concern due to about -3 psi (gage) pressure or 3 psi less than atmospheric pressure. For the large blockage, the results are truly catastrophic. All PVC pipes will fail under the conditions of Fig. 15 for $s = 10\%$. One way to correct the problem of unacceptably low pressures in the pipe is to install vacuum breakers (valves that automatically allow air into the system should the pressure become too small) at the low-pressure locations. This will be discussed in more detail below.

We can conclude that minor losses are an issue for concern especially as they affect the design of the filter system in the reservoir. We may generalize the results of this study by noting that pressure will become more negative with increases in $s$, $D$, and $z_1$. Increasing the delivery pressure, $p_2$, reduces the velocity in the pipe and thus, reduces the effect of the minor losses.

For general cases where the pipe is not straight, the pressure distribution from the energy equation needs to be solved for the specific case of the given contour of the pipe. This is normally carried out using finite differences. The equation to be solved is

$$
p(z) = \frac{p(z)}{\rho gz_1} = 1 - \frac{z}{z_1} - \frac{\nu^2}{2gD} \left\{ f(\nu, D) \frac{y(z)}{z_1} + \frac{D}{z_1} \left[ \alpha_2 + \int_{z_1}^{z} dK(z) \right] \right\}, \quad (19)
$$

where $y(z)$ is the distance from the reservoir ($z = z_1$) measured along the pipe. This function needs to be specified by the designer. The mean flow speed, $\nu$, is from the solution of eqn. (14), where $\sum_{i=1}^{N} K_i = \int_{z_1}^{0} dK(z)$ is the sum of all minor losses in the pipe system.
Figure 15: Minor loss effects on static pressure in a single straight pipe. Large blockage at entrance to pipe in reservoir. \( z/z_1 = 1 \) is located at the reservoir. \( z/z_1 = 0 \) is at the delivery location.

**Example: Straight Pipe**

Consider the example for the following system where the contour of the ground provides a uniform slope from the source to a storage tank. Because of this, we will consider the pipe to straight, i.e., it will have no bends from elbows. A flow rate measurement at the source has determined \( Q = 0.4 \) l/s and an Abney level is used to find the slope of the system of \( s = 0.008 \). An altimeter and a GPS give the elevation difference between the source and the tank (\( z_1 \)) of 64 m. Two instruments are used to find elevation since this may be one of the most uncertain of all of the measurements that characterizes the system. Normally five or more satellites are required to obtain a reliable altitude measurement from a GPS. This is difficult to achieve if there is a tree canopy that covers the source. Even with a multitude of satellites, the altitude reading from a GPS is still subject to approximately a \( \pm 10 \) m uncertainty. The uncertainty dictates that the designer consider the lowest value of \( z_1 \) rather than the actual reading. Thus, we take \( z_1 = 64 - 10 \) m = 54 m.

A filter at the inlet of the source is known to give a \( K \) value of 200 (extrapolation from Potter and Wiggert, 2002a) and there are two 90° elbows \( (L_e/D = 2 \times 30) \) entering the tank and a partially open globe valve \( (K = 300, \text{ extrapolation from Potter and Wiggert, 2002a}) \) at the inlet to the tank. The sudden enlargement from the supply pipe where it enters the tank \( (K=1) \) is negligible.

The surface of the storage tank is at atmospheric pressure \( (p_2 = 0) \) and, thus, \( F = 0 \) for this example. If we neglect minor losses for the moment, the solution for the pipe diameter is from Fig. 5 for a straight pipe \( (\lambda = 1) \) with zero delivery pressure. For the prescribed values for \( Q \) and \( s \), obtain \( D \) to be slightly less than 1.5 inch. You would specify a 1.5-inch pipe to satisfy the current conditions, or perhaps, a 2-inch pipe to accommodate an increase in the flow rate from the source in the future.
Consider the effect of the minor losses. The solution from the Mathcad sheet with the prescribed values for $L_c/D$ and $K$ as above gives $D = 1.36$ inch. This corresponds to a 1.5-inch pipe ($D = 1.61$ inch for a nominal 1.5-inch pipe). Thus, we see that the minor losses have no effect on the pipe size for this application. The values for $K$ and/or $L_c/D$ would need to increase almost 1000-fold to necessitate a 2-inch pipe size for this example. However, should the elevation head, $z_1$, decrease to a small value of 4 m, and all else in this example maintained as above, the solution shows that a 2-inch pipe size will be required. Thus, we see that special attention needs to be paid to the sizing the pipes (and storage tanks) for low-elevation-head networks.

Example: Pipe of Arbitrary Length

Here, we take the same example as above but allow the pipe to follow a contour different than the straight path from the source to the tank. For instance, this may be required when there is a blockage between the source and the tank such as a road or a house. In this case, the ratio of the actual pipe length to the one if the pipe were straight ($\lambda$) is much greater than 1. In this example, let us assume that $\lambda = 2.5$ and that there are now 20-90° elbows in the flow path ($L_c/D = 20 \times 30 = 600$). Assuming a total $K$ value of 500 as above, we obtain $D = 1.63$ inch from the Mathcad sheet. This corresponds to a 2-inch pipe ($D = 2.067$ inch for a nominal 2-inch pipe and $D = 1.61$ inch for a nominal 1.5-inch pipe). Because the diameter required to satisfy the given condition is only slightly larger than 1.5 inch (and we choose a 2-inch pipe), the volume flow rate that the 2-inch pipe can pass is $Q = 0.754$ l/s, a value nearly twice that currently produced by the source. This would accommodate plenty of additional flow rate should the source increase in the future.

A Warning: Use of the Hazen-Williams Equation

There are many tools for design employed in engineering practice in the workforce. These range from commercially available and in-house-written computer codes to formulas and nomographs (Anon., The Crane Company, 1970) for restrictive applications. One set of such formulas are of the Hazen-Williams variety. They relate the head loss (the right side of eqn. (1)) to the volume flow rate through a hydraulic resistance value (Potter and Wiggert, 2002b). One such formula is

$$H_L = RQ^\beta,$$

where $R$ is the resistance and $\beta$ is an exponent. For water flow (only), the expression for $R$ is from

$$R = \frac{KL}{C^{1/3} D^m},$$

where $\beta = 1.85$, $m = 4.87$, and $C$ is the Hazen-Williams coefficient which depends on the roughness of the pipe. To make matters a bit more cloudy, the constant $K$ depends on the system of units used, one number for SI units and a different one for English units, though it is unclear as to which units are to be used in each system. For example is the flow rate in m$^3$/s or l/s? This is uncertain without further exploration.

You can see that the Hazen-Williams formula of eqn. (20) is like a “pencil-and-paper” computer program where numbers alone are input and a numerical answer comes out; the units for both input and output are only implied. The potential downside of this type of model for head loss in a pipe system are obvious. First, one must be certain of the system of units, as would be the case with any computer program. Secondly, the user must have an understanding of the fundamentals
Figure 16: Friction factor from the accurate Colebrook equation and the not-so-accurate Hazen-Williams formula. The Swamee-Jain formula is similar to that from Haaland (White, 1999a) in eqn. (6).

The Hazen-Williams formulas are also restricted to a particular fluid in a particular temperature range, the properties for which (like density and viscosity) define one of the coefficients in eqn. (21). This narrows the usefulness of the formulas for a general case and, unless one pays attention to these restrictions, may result in erroneous calculations.

The combined effect of the above limitations produces a formula that may provide only a coarse approximation to the solution of a pipe-flow problem. In Fig. 16 (Potter and Wiggert, 2002b) we show a plot of the friction factor $f$ from several sources including the Colebrook equation (the accepted formula for friction factor) and the equivalent one from Hazen-Williams. The lack of accuracy between the these two is clear. From this figure alone, application of the results from a Hazen-Williams calculation without considerable checking of the answer would be more hazardous than the use of the Darcy-Weisbach equation. With the use of the Darcy-Weisbach equation, fluid properties and other parameters in the problem are entered with their appropriate units and the solution developed in the normal, familiar way. A computer package like Mathcad makes this approach much easier and simpler compared with paper-and-pencil, and the ability to use and cancel units in Mathcad renders calculations like flow distribution in a pipe system almost pain-free.

The bottom line in this saga is the following. Where possible, stick to the fundamentals in everything that you do, fluid dynamics and pipe-flow calculations included. Do not develop unnecessarily restrictive formulas that you might commit to memory which could surface for your use in an inappropriate application. For the pipe flow calculations that you make in your work at Villanova, there is no good reason to use the Hazen-Williams formulas. However, please be aware that for a number of very good reasons, economy included, the industry uses restrictive formulas every day, including the Hazen-Williams variety. This also includes The Blue Book which you will
use for your further study of gravity-driven water systems.

**Graphical Interpretations: Hydraulic Grade Line (HGL) and Energy Grade Line (EGL)**

Hydraulicists (persons who carry out calculations for pipe and open-channel fluid flows) sometimes re-write eqn. (1) by dividing both sides by \(g\) to obtain

\[
\frac{p(z)}{\rho g} + \frac{\alpha_2 \bar{u}(z)^2}{2g} + z = h_T - h_L(z),
\]

where \(h_T\) is the total head at \(z = z_1\) and \(h_L(z)\) is the \(z\)-dependent head loss due to frictional energy losses, both major and minor. The relationship between \(h\) and \(H\) (see eqn. (1)) is \(gh = H\). Since eqn. (22) must be dimensionally homogeneous, each term on the left side is a height, or head, above \(z = 0\). Specifically, \(p(z)/\rho g\) is the “pressure head” due to the static pressure \(p(z)\), \(\alpha_2 \bar{u}(z)^2/2g\) is the “velocity head,” and \(z\) is the “elevation head.” We see from eqn. (22) that the three heads must add up to the total head less that due to energy losses. A plot of \(h_T - h_L(z)\), or alternately, \(p(z)/\rho g + \alpha_2 \bar{u}(z)^2/(2g) + z\), vs. horizontal distance over which the pipe runs produces on the vertical axis an “Energy Grade Line” or EGL. For a lossless flow (inviscid, for which the energy equation becomes the Bernoulli equation), \(h_L = 0\) and the EGL is a constant height, \(h_T\). Otherwise, the EGL has a negative slope for real flows, the value of which depends on the major frictional loss and the distribution and magnitude of the minor loss coefficients. For example, a minor loss will result in a sudden (i.e., vertical) reduction in the EGL height whereas the major loss affects only the slope of the EGL over some finite distance.

The “Hydraulic Grade Line” or HGL is defined as \(p(z)/\rho g + z\) and thus includes only the pressure and elevation heads. For a horizontal pipe, the HGL height is a measure of the pressure distribution in the pipe and for a real flow, will have a negative slope due to the frictional losses. The difference between the EGL and the HGL is a measure of the velocity in the pipe. For a pipe of uniform diameter, the velocity is constant and the difference between the EGL and HGL lines will be constant, i.e., the slopes of each line will be the same. Minor losses will affect the HGL in the same manner as they do the EGL; a minor loss resulting in a sudden drop in the HGL due to loss of static pressure.

**Extension to a Multiple-Pipe System**

The above developments may be extended to a network having more than a single pipe as discussed in this section. For a system having any number of pipes of any diameter, connected in any manner, the solution for the pipe diameters is found by:

1. Applying the energy equation for pipe flow, eqn. (1), to each leg of the network for which there is a constant pipe diameter,
2. At each junction in the network, mass conservation requires that the sum of the volume flow rates into the junction must be zero. For example, if there are \(n\) pipes at a junction, mass conservation requires \(\sum_{i=1}^{n} Q_i = 0\).
In writing mass conservation in this way, please be aware that inflows to the junction are assumed to have positive numerical values, and outflows have negative numerical values.

After writing the equations as directed by this procedure, the following conditions are used to simplify the equations before carrying out the solution.

1. For all reservoir and open-tank surfaces, \( p = 0 \) and \( \pi = 0 \).

2. For a faucet valve, the pipe diameters leading to the tap stand and away from it are obviously selected based on flow conditions and not a shut valve. When the faucet valve is open, the static pressure at the value outlet is zero (atmospheric pressure or zero gage pressure) and there will be the need for a minor loss for the valve, especially if it is only partly open. If the valve is shut, the pipe diameters have no effect on the pressure in the system, since the pressure at any point arises from only hydrostatics. That is, \( p = \rho g (z_1 - z) \).

3. At all junctions, one can imagine the mixing of flows from/to multiple pipes as taking place in a small mixing box. In this box, the pressure is equal everywhere. Thus, at all junctions, \( p \) is the same for all pipes at that location. In addition, one may approximate the velocity in this mixing box as small compared with the velocity in a pipe (\( \pi = 0 \) at a junction). The latter is a conservative assumption that simplifies the formulation of the problem.

Consider the 3-pipe system as shown in Fig. 17. The pipes are labeled a, b, and c, and each has a mean flowspeed, \( \bar{u} \), volume flow rate, \( Q \), diameter, \( D \), and length measured along the path of the pipe, \( L \). The change in elevation between the top and bottom of each pipe is \( \Delta z \). For example, \( \Delta z_a \) is the elevation change between the top and bottom of pipe a, where the bottom of this pipe is located where the pressure is \( p_2 \).

The energy equation is, in general,

\[
(p_1 + \alpha_1 \frac{\pi_1^2}{2} + g z_1) - (p_2 + \alpha_2 \frac{\pi_2^2}{2} + g z_2) = C_L \frac{\pi^2}{2} = [f(\bar{u}, D)(\frac{L}{D}) + \sum_{i=1}^{M} \frac{I_e}{D} |_{i} + \sum_{i=1}^{N} K_i \frac{\pi^2}{2}],
\]

where, for compactness, we have defined \( C_L \) as a loss coefficient that includes the major and minor loss terms. Please note that \( C_L \) for each pipe depends on \( \bar{u} \) and \( D \) for the pipe (through the Reynolds number), and the values of the minor loss coefficients, \( K \) and \( L_e/D \).
Equation (23) is written for pipes a, b, and c to get

\[ p_2 = \rho g \Delta z_a + \frac{\rho}{2} (-C_{L,a} + \alpha) u_a^2, \]
\[ p_2 = -\rho g \Delta z_b + \frac{\rho}{2} (C_{L,b} + \alpha) u_b^2, \]
\[ p_2 = -\rho g \Delta z_c + \frac{\rho}{2} (C_{L,c} + \alpha) u_c^2. \]

The continuity equation for each pipe relates \( Q \) to \( u \) through

\[ u = \frac{4Q}{\pi D^2}. \]

Substituting this into eqn. (24), obtain

\[ p_2 = \rho g \Delta z_a + \frac{\rho}{2} (-C_{L,a} + \alpha) \left(\frac{4Q_a}{\pi D_a^2}\right)^2, \]
\[ p_2 = -\rho g \Delta z_b + \frac{\rho}{2} (C_{L,b} + \alpha) \left(\frac{4Q_b}{\pi D_b^2}\right)^2, \]
\[ p_2 = -\rho g \Delta z_c + \frac{\rho}{2} (C_{L,c} + \alpha) \left(\frac{4Q_c}{\pi D_c^2}\right)^2. \]

The numerical values appearing in Table 1 apply to this problem. By choosing a pressure, \( p_2 \), eqn. (25) is solved with the continuity equation

\[ Q_a + Q_b + Q_c = 0, \]

to obtain diameters \( D_a, D_b, \) and \( D_c \). The solution, obtained in Mathcad using the Given...Find construct, is presented in Fig. 18 for a wide range of values for \( p_2 \). We see that the diameters for pipes a and c are not very sensitive to the pressure, \( p_2 \), at the delivery point (i.e., the mixing box). \( D_a \) is approximately 1 inch and \( D_c \) is about 0.75 inch. On the other hand, \( D_c \) is very sensitive to \( p_2 \), especially at low values of \( p_2 \) where the driving force for the flow in pipe c approaches the elevation head of \( \Delta z_c = -1 \) m, (note that the negative value for \( \Delta z_c \) means that there is an elevation increase from the delivery point to the faucet valve). To be conservative, and to allow for an expansion in the flow rates in the future, the final selections for pipe diameters \( D_a, D_b, \) and \( D_c \) may be 1 inch throughout, with a design pressure of about 3-5 psi for \( p_2 \).

We may generalize the equations for the case of an \( n \)-pipe network connected in a manner as shown in Fig. 17. The energy equation for pipe a remains as it is written as the first of eqn. (25). For every other pipe in the network, the form is identical to the remaining two equations in eqn. (25). That is, for the \( j \)th pipe in the network we can write the energy equation as

\[ p_2 = -\rho g \Delta z_j + \frac{\rho}{2} (C_{L,j} + \alpha) \left(\frac{4Q_j}{\pi D_j^2}\right)^2. \]

For an \( n \)-pipe network, the continuity equation is written as

\[ \sum_{i=1}^{n} Q_i = 0. \]
Figure 18: The diameters for the case of a 3-pipe network.

The solution of the resulting \( n \) simultaneous non-linear algebraic equations in terms of \( D_a, D_b, D_c \ldots D_n \) is then obtained (say, in Mathcad) once a value for \( p_2 \) is prescribed.

**Other Considerations in the Design**

To be continued...

**Case Study**

Consider the example for...
Nomenclature

\begin{itemize}
\item \( A \) \quad \text{area, m}^2
\item \( C \) \quad \text{coefficient}
\item \( D \) \quad \text{inside diameter of pipe, m}
\item \( e \) \quad \text{roughness of pipe wall, m}
\item \( f \) \quad \text{Moody friction factor}
\item \( F \) \quad \text{dimensionless pressure at delivery location, } \frac{p_2}{\rho g z_1}
\item \( g \) \quad \text{gravitational acceleration, m/s}^2
\item \( K \) \quad \text{minor loss coefficient}
\item \( H \) \quad \text{head, (m/s)}^2
\item \( h \) \quad \text{head, m}
\item \( L \) \quad \text{pipe length, m}
\item \( \ell \) \quad \text{pipe length from inlet to outlet measured in horizontal plane, m}
\item \( N \) \quad \text{number of minor-loss elements accounted for using } K \text{ values}
\item \( n \) \quad \text{number of pipes in a multiple-pipe network}
\item \( M \) \quad \text{number of minor-loss elements accounted for using } \frac{L_e}{D} \text{ values}
\item \( p \) \quad \text{static pressure, Pa}
\item \( Q \) \quad \text{volume flow rate, liters/s}
\item \( \text{Re} \) \quad \text{Reynolds number}
\item \( \overline{\pi} \) \quad \text{cross-sectional average flow speed, m/s}
\item \( x, z \) \quad \text{horizontal and vertical coordinates, m}
\item \( y \) \quad \text{distance from inlet of pipe measured along the pipe path, m}
\item \( \alpha \) \quad \text{kinetic energy correction factor}
\item \( \lambda \) \quad \text{factor to calculate pipe length from elevation and run lengths}
\item \( \nu \) \quad \text{kinematic viscosity of fluid, m}^2/\text{s}
\item \( \rho \) \quad \text{density of fluid, kg/m}^3
\end{itemize}

Subscripts

\begin{itemize}
\item \( 1, 2 \) \quad \text{inlet and outlet states}
\item \( e \) \quad \text{equivalent}
\item \text{entry} \quad \text{at entry of pipe}
\item \( i \) \quad \text{index}
\item \( L \) \quad \text{loss}
\item \( T \) \quad \text{total}
\end{itemize}

References


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